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Zeros of Sobolev orthogonal polynomials following from coherent pairs

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Abstract

Let $\{S_n^\lambda\}$ denote the monic orthogonal polynomial sequence with respect to the Sobolev inner product

$$\langle f,g\rangle_S = \int_{-\infty}^{\infty} fg\,\mathrm{d}\psi_0 + \lambda\int_{-\infty}^{\infty} f'g'\,\mathrm{d}\psi_1,$$

where $\{\mathrm{d}\psi_0,\mathrm{d}\psi_1\}$ is a so-called coherent pair and $\lambda>0$. Then S_n^λ has n different, real zeros. The position of these zeros with respect to the zeros of other orthogonal polynomials (in particular Laguerre and Jacobi polynomials) is investigated. Coherent pairs are found where the zeros of S_{n-1}^λ separate the zeros of S_n^λ. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we investigate polynomials orthogonal with respect to a Sobolev inner product of the form

$$\langle f,g\rangle_S = \int_a^b f(x)g(x)\,\mathrm{d}\psi_0(x) + \lambda\int_a^b f'(x)g'(x)\,\mathrm{d}\psi_1(x), \tag{1.1}$$

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where $\lambda > 0$. Let $\{P_n\}$ and $\{Q_n\}$ denote the monic orthogonal polynomial sequences (MOPS) with respect to $\mathrm{d}\psi_0$, respectively $\mathrm{d}\psi_1$. The pair $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ is called a coherent pair if there exist nonzero constants D_n such that

$$Q_n(x) = \frac{P'_{n+1}(x)}{n+1} + D_n \frac{P'_n(x)}{n}, \quad n \geqslant 1.$$

The concept of coherent pair was introduced by Iserles et al. in [1]. It proved to be very fruitful in the research of Sobolev orthogonal polynomials. In this paper we assume $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ to be a coherent pair. Let $\{S_n^\lambda\}$ denote the MOPS with respect to the inner product (1.1). We will investigate the location of the zeros of S_n^λ.

In [4] it already has been proved that for $n \geqslant 2$ and λ *sufficiently large*, S_n^λ has n different, real zeros separated by the zeros of P_{n-1} and by the zeros of Q_{n-1}; at most one of the zeros of S_n^λ is outside (a,b). Marcellán et al. [2] studied the special case $(a,b) = (0,\infty), \mathrm{d}\psi_0 = \mathrm{d}\psi_1 = x^\alpha e^{-x}\,\mathrm{d}x$, $(\alpha > -1)$, and proved that for $n \geqslant 2$ and *all* $\lambda > 0$ the polynomial S_n^λ has n real zeros, interlacing with those of L_n^α; if $\alpha \geqslant 0$ all zeros of S_n^λ are positive, if $-1 < \alpha < 0$, then S_n^λ may have one negative zero.

In [5] a complete classification of all coherent pairs has been given. This classification enables us to generalize the above mentioned results. We will prove in Section 4 (Theorem 4.1): if $n \geqslant 2$, then for *all* $\lambda > 0$ the polynomial S_n^λ has n real zeros; the zeros of S_n^λ interlace with the zeros of P_n and are separated by the zeros of P_{n-1}.

For further research the coherent pairs have to be divided into three classes (type A, B and C) depending on the regularity of $\mathrm{d}\psi_0$ and $\mathrm{d}\psi_1$. For coherent pairs of type A and B, the more regular cases, we can derive more information on the position of the zeros of S_n^λ. We prove in Section 4: if $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ denote a coherent pair of type A or B and $n \geqslant 2$, then for *all* $\lambda > 0$ the zeros of Q_{n-1} separate the zeros of S_n^λ (Theorems 4.2(iii) and 4.3(ii)); moreover for *all* $\lambda > 0$ the zeros of S_{n-1}^λ separate the zeros of S_n^λ (Theorem 4.5).

In Section 5 we show that the last mentioned assertions are, in general, not satisfied by the zeros of S_n^λ for all $\lambda > 0$ when $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ is a coherent pair of type C.

In particular we investigate the case $(a,b) = (0,\infty), \mathrm{d}\psi_0 = \mathrm{d}\psi_1 = x^\alpha e^{-x}\,\mathrm{d}x$ also studied in [2]. If $\alpha > 0$, then $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ is of type A, with $\alpha = 0$ it is of type B and for $-1 < \alpha < 0$ we have a coherent pair of type C. We show (Theorem 5.1) that if $-1 < \alpha < 0$ and $n \geqslant 3$, then there exist λ such that the zeros of S_{n-1}^λ do not separate the zeros of S_n^λ. There even exists a λ such that S_{n-1}^λ and S_n^λ have a zero in common.

In Section 2 we derive some auxiliary results on S_n^λ and we give the classification of the coherent pairs. In Section 3 we derive the Lemmas 3.5 and 3.6; they constitute the basis for the proofs of the results in Sections 4 and 5.

2. Coherent pairs

Consider the inner product

$$\langle f, g\rangle_S = \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}\psi_0(x) + \lambda \int_{-\infty}^{\infty} f'(x)g'(x)\,\mathrm{d}\psi_1(x), \tag{2.1}$$

where $d\psi_0$ and $d\psi_1$ are measures on the real line and $\lambda > 0$. Let $\{P_n\}$ denote the monic orthogonal polynomial sequence (MOPS) with respect to $d\psi_0$ and $\{Q_n\}$ the MOPS with respect to $d\psi_1$. We assume that $\{d\psi_0, d\psi_1\}$ is a coherent pair, i.e. there exist non-zero constants D_n such that

$$Q_n = \frac{P'_{n+1}}{n+1} + D_n \frac{P'_n}{n}, \quad n \geqslant 1. \tag{2.2}$$

In [5] the coherent pairs are completely determined. For the research on the zeros it is convenient to divide the coherent pairs in three classes of increasing irregularity given in the following scheme. The scheme mentions all coherent pairs apart from a linear change in the variable. Observe that the three classes are not completely disjoint.

Type A

(i) $(a,b) = (0,\infty), d\psi_0 = (x-\xi)x^\alpha e^{-x}\,dx, d\psi_1 = x^{\alpha+1}e^{-x}\,dx$, with $\xi \leqslant 0,\ \alpha > -1$.
(ii) $(a,b) = (-1,1), d\psi_0 = (x - \xi)(1 - x)^\alpha(1 + x)^\beta\,dx, d\psi_1 = (1 - x)^{\alpha+1}(1 + x)^{\beta+1}\,dx$, with $\xi \leqslant -1, \alpha > -1,\ \beta > -1$.

Type B

(iii) $(a,b) = (0,\infty), d\psi_0 = e^{-x}\,dx + M\delta(0), d\psi_1 = e^{-x}\,dx$, with $M \geqslant 0$.
(iv) $(a,b) = (-1,1), d\psi_0 = (1-x)^\alpha\,dx + M\delta(-1), d\psi_1 = (1-x)^{\alpha+1}\,dx$, with $\alpha > -1, M \geqslant 0$.

Type C

(v) $(a,b) = (0,\infty), d\psi_0 = x^\alpha e^{-x}\,dx, d\psi_1 = x^{\alpha+1}e^{-x}/(x-\xi)\,dx + M\delta(\xi)$, with $\xi \leqslant 0,\ \alpha > -1,\ M \geqslant 0$.
(vi) $(a,b) = (-1,1), d\psi_0 = (1-x)^\alpha(1+x)^\beta\,dx, d\psi_1 = (1-x)^{\alpha+1}(1+x)^{\beta+1}/(x-\xi)\,dx + M\delta(\xi)$, with $\xi \leqslant -1,\ \alpha > -1,\ \beta > -1,\ M \geqslant 0$.

We will denote the continuous part of $d\psi_0$ by $w_0\,dx$ and the continuous part of $d\psi_1$ by $w_1\,dx$. In all cases the continuous parts are restricted to the interval (a,b).

It is convenient for coherent pairs of type B also to define a ξ; we put $\xi = 0$ in case (iii) and $\xi = -1$ in case (iv).

We will refer to (i), (iii) and (v) where $(a,b) = (0,\infty)$ as Laguerre cases; (ii), (iv) and (vi) with $(a,b) = (-1,1)$ will be called the Jacobi situation.

We often use the Laguerre polynomials L_n^α and Jacobi polynomials $P_n^{\alpha,\beta}$. Their monic versions are denoted by $\hat{L}_n^\alpha$ and $\hat{P}_n^{\alpha,\beta}$ (trivially they have the same zeros as the nonmonic versions). The monic polynomials satisfy

$$\frac{d}{dx}\hat{L}_n^\alpha = n\hat{L}_{n-1}^{\alpha+1}, \qquad \frac{d}{dx}\hat{P}_n^{\alpha,\beta} = n\hat{P}_{n-1}^{\alpha+1,\beta+1} \tag{2.3}$$

(compare Szegő [6], p. 102, 72). With (2.3) it is easy to check that the above mentioned pairs $\{d\psi_0, d\psi_1\}$ are indeed coherent pairs (see [3] or [5]). Moreover it follows that the constants D_n in (2.2) satisfy

$$D_n > 0, \quad n \geqslant 1. \tag{2.4}$$

Let $\{S_n^\lambda\}$ denote the MOPS with respect to the inner product (2.1). Obviously $S_0^\lambda = P_0 = 1,\ S_1^\lambda = P_1$. When there is no risk of confusion we omit the λ and write S_n.

Lemma 2.1. *For $n \geqslant 1$ the following relations are satisfied*

$$\frac{P_{n+1}}{n+1} + D_n \frac{P_n}{n} = \frac{S_{n+1}^{\lambda}}{n+1} + d_n \frac{S_n^{\lambda}}{n}, \tag{2.5}$$

$$Q_n = \frac{(S_{n+1}^{\lambda})'}{n+1} + d_n \frac{(S_n^{\lambda})'}{n}, \tag{2.6}$$

where

$$d_n = \frac{\int P_n^2 \, \mathrm{d}\psi_0}{\langle S_n^{\lambda}, S_n^{\lambda} \rangle_S} D_n > 0. \tag{2.7}$$

Proof. Since $\{S_n^{\lambda}\}$ is a basis for the polynomials we can write

$$\frac{P_{n+1}}{n+1} + D_n \frac{P_n}{n} = \frac{S_{n+1}^{\lambda}}{n+1} + \sum_{k=0}^{n} c_k^{(n)} S_k^{\lambda},$$

for some constants $c_k^{(n)}$. Then it follows from (2.1) and (2.2)

$$c_k^{(n)} \langle S_k^{\lambda}, S_k^{\lambda} \rangle_S = \int \left(\frac{P_{n+1}}{n+1} + D_n \frac{P_n}{n} \right) S_k^{\lambda} \, \mathrm{d}\psi_0 + \lambda \int Q_n (S_k^{\lambda})' \, \mathrm{d}\psi_1.$$

Hence $c_k^{(n)} = 0$ if $0 \leqslant k \leqslant n-1$ and

$$c_n^{(n)} \langle S_n^{\lambda}, S_n^{\lambda} \rangle_S = \frac{D_n}{n} \int P_n S_n^{\lambda} \, \mathrm{d}\psi_0 = \frac{D_n}{n} \int P_n^2 \, \mathrm{d}\psi_0,$$

which proves (2.5). Relation (2.6) follows by differentiation of (2.5) and using (2.2). □

2.1. The polynomial S_n^{∞}

Observe $\langle S_n^{\lambda}, S_n^{\lambda} \rangle_S > \lambda \int (S_n^{\lambda\prime})^2 \, \mathrm{d}\psi_1$. Then, by the minimal property,

$$\langle S_n^{\lambda}, S_n^{\lambda} \rangle_S > \lambda n^2 \int Q_{n-1}^2 \, \mathrm{d}\psi_1,$$

and (2.7) implies $\lim_{\lambda \to \infty} d_n = 0$, $n \geqslant 1$.

Since $S_1^{\lambda} = P_1$, we obtain from (2.5) by induction that

$$S_n^{\infty} = \lim_{\lambda \to \infty} S_n^{\lambda} \quad (n \geqslant 2)$$

exists and

$$\frac{S_n^{\infty}}{n} = \frac{P_n}{n} + D_{n-1} \frac{P_{n-1}}{n-1}, \quad n \geqslant 2. \tag{2.8}$$

From (2.5) and (2.8) we obtain

$$\frac{S_n^{\infty}}{n} = \frac{S_n^{\lambda}}{n} + d_{n-1}\frac{S_{n-1}^{\lambda}}{n-1}, \quad n \geqslant 2. \tag{2.9}$$

In particular (2.8) and (2.2) imply

$$\int S_n^{\infty}\,\mathrm{d}\psi_0 = 0, \quad n \geqslant 2. \tag{2.10}$$

$$(S_n^{\infty})' = nQ_{n-1}, \quad n \geqslant 2. \tag{2.11}$$

The relations (2.10) and (2.11) completely determine S_n^{∞}.

For coherent pairs of type A it follows from (2.3), (2.10) and (2.11) that $S_n^{\infty} = \hat{L}_n$ in the Laguerre case and $S_n^{\infty} = \hat{P}_n^{\alpha,\beta}$ in the Jacobi case ($n \geqslant 2$).

For coherent pairs of type B the S_n^{∞} are given by ($n \geqslant 2$):

$S_n^{\infty}(x) = x\hat{L}_{n-1}^{(1)}(x)$ in the Laguerre case,

$S_n^{\infty}(x) = (x+1)\hat{P}_{n-1}^{\alpha,1}(x)$ in the Jacobi case.

It can be checked that the polynomials mentioned indeed satisfy (2.10) and (2.11). (To check (2.11) take the $\mathrm{d}\psi_1$ inner product with x^k, $k \in \{0, 1, \ldots, n-1\}$ and apply integration by parts.)

2.2. The polynomial S_n^0

Since $S_1^{\lambda} = P_1$ we have with (2.7) $\lim_{\lambda\downarrow 0} d_1 = D_1$.

Then induction on (2.5) and (2.7) gives

$$S_n^0 = \lim_{\lambda\downarrow 0} S_n^{\lambda} = P_n, \quad n \geqslant 2 \tag{2.12}$$

$$\lim_{\lambda\downarrow 0} d_n = D_n, \quad n \geqslant 1.$$

3. The moments

We introduce the moments

$$m_i^n = \int_a^b S_n(x)(x-\xi)^i w_0(x)\,\mathrm{d}x.$$

For coherent pairs of type A we take $i \in \{-1, 0, 1, \ldots\}$ and for coherent pairs of type B and C we take $i \in \{0, 1, 2, \ldots\}$.

Moreover we will use the moments

$$\mu_i^n = \int_{-\infty}^{\infty} S_n'(x)(x-\xi)^i\,\mathrm{d}\psi_1(x) \tag{3.1}$$

for all types we take $i \in \{0, 1, 2, \ldots\}$.

Since for $1 \leqslant i \leqslant n-1$, we have $\langle S_n, (x-\xi)^i \rangle_S = 0$ the moments are related by

$$m_i^n = -\lambda i \mu_{i-1}^n, \quad i = 1, 2, \ldots, n-1. \tag{3.2}$$

Other relations between μ_i^n and m_i^n will be constructed from the following lemma.

Lemma 3.1. 1. *Laguerre case. For $i \geqslant 0$, $\alpha > -1$ it holds*

$$\begin{aligned}
&\int_0^\infty S_n'(x)(x-\xi)^i x^{\alpha+1} e^{-x}\,\mathrm{d}x \\
&\quad = \int_0^\infty S_n(x)(x-\xi)^{i+1} x^\alpha e^{-x}\,\mathrm{d}x - (i+1+\alpha-\xi)\int_0^\infty S_n(x)(x-\xi)^i x^\alpha e^{-x}\,\mathrm{d}x \\
&\qquad - i\xi \int_0^\infty S_n(x)(x-\xi)^{i-1} x^\alpha e^{-x}\,\mathrm{d}x.
\end{aligned}$$

2. *Jacobi case. For $i \geqslant 0$, $\alpha > -1$, $\beta > -1$ it holds*

$$\begin{aligned}
&\int_{-1}^{+1} S_n'(x)(x-\xi)^i (1-x)^{\alpha+1}(1+x)^{\beta+1}\,\mathrm{d}x \\
&\quad = (\alpha+\beta+i+2)\int_{-1}^{+1} S_n(x)(x-\xi)^{i+1}(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x \\
&\qquad + \{(\alpha+1)(\xi+1)+(\beta+1)(\xi-1)+2i\xi\}\int_{-1}^{+1} S_n(x)(x-\xi)^i(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x \\
&\qquad + i(\xi^2-1)\int_{-1}^{+1} S_n(x)(x-\xi)^{i-1}(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x.
\end{aligned}$$

For $i = 0$ the third term in the right-hand side of the relations is omitted.

The lemma can be proved by integration by parts, where the constant terms vanish. Observe that the coefficients before the three integrals in the right-hand sides of the relations are, respectively, positive, negative and nonnegative.

It is our intention to determine the signs of m_i^n.

Lemma 3.2. *All coherent pairs satisfy*:
if $n \geqslant 1$, then $\operatorname{sgn} \mu_0^n = (-1)^{n-1}$;
if $n \geqslant 2$, then $\operatorname{sgn} m_1^n = (-1)^n$.

Proof. Obviously $\mu_0^1 = \int S_1'\,\mathrm{d}\psi_1 = \int \mathrm{d}\psi_1 > 0$.

From (2.6) it follows that for $n \geqslant 1$ the moments μ_0^n and μ_0^{n+1} have opposite sign. Then the first result follows. The second assertion follows from (3.2). □

Lemma 3.3. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type B. Let $n \geqslant 2$. Then*

$$sgn\, S_n^{\lambda}(a) = (-1)^n,$$

$$m_0^n = 0 \text{ or } sgn\, m_0^n = (-1)^{n-1}.$$

Proof. From $\langle S_n, 1\rangle_S = 0$ it follows

$$m_0^n = -S_n(a)M. \tag{3.3}$$

Integrating by parts we obtain

$$\mu_0^n = \int_a^b S_n'(x)w_1(x)\,dx = -S_n(a)w_1(a) - \int_a^b S_n(x)w_1'(x)\,dx.$$

Observe $w_1' = -cw_0$ with $c = 1$ in the Laguerre case and $c = \alpha + 1 > 0$ in the Jacobi case. Hence, with (3.3),

$$\mu_0^n = -S_n(a)w_1(a) + cm_0^n = -S_n(a)(w_1(a) + cM).$$

Then Lemma 3.2 implies

$$sgn\, S_n(a) = (-1)^n,$$

and with (3.3),

$$m_0^n = 0 \text{ if } M = 0, \quad sgn\, m_0^n = (-1)^{n-1} \text{ if } M \neq 0. \qquad \square$$

Lemma 3.4. *All coherent pairs satisfy for $n \geqslant 2$:*

$$m_0^n = 0 \text{ or } sgn\, m_0^n = (-1)^{n-1};$$

$$sgn\, m_i^n = (-1)^{n+i-1}, \quad i = 1, \dots, n-1.$$

Coherent pairs of type A moreover satisfy for $n \geqslant 2$:

$$sgn\, m_{-1}^n = (-1)^n.$$

Proof. For coherent pairs of type A or C the relation $\langle S_n, 1\rangle_S = 0$ implies $m_0^n = 0$. Together with Lemma 3.3 this proves the first assertion. For the determination of the signs of the other moments we have to distinguish between type A and types B or C.

(a) Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A. Lemma 3.1 with $i = 0$ gives

$$\mu_0^n = a_0 m_0^n - b_0 m_{-1}^n = -b_0 m_{-1}^n$$

for some $a_0 > 0$ and $b_0 > 0$. Then Lemma 3.2 gives $sgn\, m_{-1}^n = (-1)^n$.

For $i \geqslant 1$ we obtain from Lemma 3.1

$$\mu_i^n = a_i m_i^n - b_i m_i^n + c_i m_{i-2}^n \tag{3.4}$$

for some $a_i > 0$, $b_i > 0$ and $c_i \geqslant 0$.

Using (3.4) and (3.2) we can determine subsequently the signs of $\mu_1^n, m_2^n, \mu_2^n, \dots, m_{n-1}^n$.

(b) Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type B or C. For $i \geqslant 1$ we write

$$\mu_i^n = \int_a^b S_n'(x)(x-\xi)^{i-1}\{(x-\xi)w_1(x)\}\,dx.$$

Now we apply Lemma 3.1 with i replaced by $i-1$.
We obtain

$$\mu_i^n = a_1 m_1^n - b_1 m_0^n \tag{3.5}$$

and for $i \geqslant 2$

$$\mu_i^n = a_i m_i^n - b_i m_{i-1}^n + c_i m_{i-2}^n \tag{3.6}$$

for some $a_i > 0$, $b_i > 0$, $c_i \geqslant 0$, $(i \geqslant 1)$.

With (3.5), Lemma 3.2 and the first assertion of the present lemma we obtain $sgn\,\mu_1^n = (-1)^n$. Proceeding as before (3.2) and (3.6) enables us to determine subsequently the signs of $m_2^n, \mu_2^n, \dots, m_{n-1}^n$. □

Remark 3.1. Define for $n \geqslant 2$ the moments $\tilde{m}_i^n$ by

$$\tilde{m}_i^n = \int_{-\infty}^{\infty} S_n(x)(x-\xi)^i\,d\psi_0,$$

where i is in the same range as in m_i^n.

Obviously $\tilde{m}_i^n = m_i^n$ with exception of $\tilde{m}_0^n$ for coherent pairs of type B with $M \neq 0$, where $\tilde{m}_0^n = 0$ and $m_0^n \neq 0$ (compare Lemma 3.3). Then $\tilde{m}_i^n$ also satisfies the assertions of Lemma 3.4.

Lemma 3.5. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A, B or C. Let $n \geqslant 2$. Let $\pi(x)$ denote a monic polynomial of degree k, $1 \leqslant k \leqslant n-1$ such that all zeros of π are real and $\geqslant \xi$. Put*

$$I = \int_a^b S_n^{\lambda}(x)\pi(x)w_0(x)\,dx, \tag{3.7}$$

$$\tilde{I} = \int_{-\infty}^{\infty} S_n^{\lambda}(x)\pi(x)\,d\psi_0(x). \tag{3.8}$$

Then $sgn\,I = sgn\,\tilde{I} = (-1)^{n+k-1}$.

Proof. Let $\xi + t_1, \ldots, \xi + t_k$ denote the zeros of π. By assumption $t_1, \ldots, t_k$ are nonnegative. Then

$$\pi(x) = (x - \xi - t_1) \ldots (x - \xi - t_k) = \sum_{i=0}^{k} c_i (x - \xi)^i,$$

where $c_k = 1$ and if $c_i \neq 0$, then $sgn\, c_i = (-1)^{k-i}$. Then

$$I = \sum_{i=0}^{k} c_i m_i^n, \qquad \tilde{I} = \sum_{i=0}^{k} c_i \tilde{m}_i^n$$

and the result follows from Lemma 3.4 and Remark 3.1. □

For coherent pairs of type A also m_{-1}^n exists so we can prove more.

Lemma 3.6. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A. Let $n \geqslant 2$ and let $\pi(x)$ denote a monic polynomial of degree k, $2 \leqslant k \leqslant n$ such that all zeros of π are real and $\geqslant \xi$. Put*

$$I = \int_a^b S_n^\lambda(x) \pi(x) \frac{w_0(x)}{x - \xi}\, dx. \tag{3.9}$$

Then $sgn\, I = (-1)^{n+k}$.

The proof is similar to the proof of Lemma 3.5. If we substitute $\pi(x) = (x - \xi)\pi_1(x)$ in Lemma 3.6 we obtain Lemma 3.5.

4. Location of the zeros

In this section we prove that S_n^λ has n different real zeros and we determine the position of these zeros with respect to the position of the zeros of other (well-known) orthogonal polynomials. The following theorem states all the location properties for which we can prove that they are satisfied by S_n^λ for all coherent pairs and all λ. It improves a result from [4].

Theorem 4.1. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A, B or C. Let $\{P_n\}$ denote the MOPS with respect to $d\psi_0$. Let $n \geqslant 2$. Then*

(i) *S_n^λ has n different, real zeros; at most one of them is outside (a, b);*

(ii) *the zeros of S_n^λ interlace with those of P_n: if $p_{1n} < p_{2n} < \cdots < p_{nn}$ denote the zeros of P_n and $s_{1n}^\lambda < s_{2n}^\lambda < \cdots < s_{nn}^\lambda$ the zeros of S_n^λ, then $s_{1n}^\lambda < p_{1n} < s_{2n}^\lambda < \cdots < s_{nn}^\lambda < p_{nn}$;*

(iii) *the $n - 1$ zeros of P_{n-1} separate the zeros of S_n^λ.*

Proof. Let η denote a zero of P_n. Put

$$\pi(x) = \frac{P_n(x)}{x - \eta}$$

in relation (3.8) of Lemma 3.5. Then

$$\int_{-\infty}^{\infty} S_n^{\lambda}(x)\frac{P_n(x)}{x-\eta}\,\mathrm{d}\psi_0(x) > 0.$$

Apply Gauss-quadrature on the zeros of P_n, then

$$S_n^{\lambda}(\eta)P_n'(\eta) > 0, \quad \eta \text{ zero of } P_n.$$

Since P_n' has opposite sign in two consecutive zeros of P_n, the same holds for S_n. This implies that S_n has at least one zero in each of the $n-1$ intervals $(p_{i-1,n}, p_{in})$, $i=2,\ldots,n$. Moreover, since P_n is monic, $sgn\, P_n'(p_{1n})=(-1)^{n-1}$, thus $sgn\, S_n(p_{1n})=(-1)^{n-1}$ and since S_n is monic too, S_n has a zero in $(-\infty, p_{1n})$. This implies part (ii) of the theorem.

Since $p_{1n},\ldots,p_{nn}$ are in (a,b) at most s_{1n}^{λ} is outside (a,b). This completes the proof of part (i) of the theorem.

In order to prove the last assertion we put

$$\pi(x)=\frac{P_{n-1}(x)}{x-\eta},$$

where η denotes a zero of P_{n-1} in (3.8). Then

$$\int_{-\infty}^{\infty} S_n^{\lambda}(x)\frac{P_{n-1}(x)}{x-\eta}\,\mathrm{d}\psi_0(x) < 0.$$

We apply Gauss-quadrature on the $n-1$ zeros of P_{n-1}. Since the degree of the integrand is $2n-2$ we have to use the quadrature formula with remainder-term (see [6], p. 378). Since all polynomials are monic the remainder-term equals $\int P_{n-1}^2\,\mathrm{d}\psi_0 > 0$.

Hence

$$S_n^{\lambda}(\eta)P_{n-1}'(\eta) < 0, \quad \eta \text{ zero of } P_{n-1}.$$

As before this implies that S_n has at least one zero between two consecutive zeros of P_{n-1}. Moreover $P_{n-1}'(p_{n-1,n-1}) > 0$, hence $S_n(p_{n-1,n-1}) < 0$ and S_n has a zero in $(p_{n-1,n-1},\infty)$. Finally $sgn\, P_{n-1}'(p_{1,n-1})=(-1)^{n-2}$, thus $sgn\, S_n(p_{1,n-1})=(-1)^{n-1}$ and S_n has a zero in $(-\infty, p_{1,n-1})$. Then the result follows. □

4.1. Remarks on Theorem 4.1

(i) S_n can have at most one zero outside (a,b) in $(-\infty,a)$. Theorem 4.2 will imply that for coherent pairs of type A all zeros of S_n are in (a,b). For coherent pairs of type B we have, by Lemma 3.3, $sgn\, S_n(a)=(-1)^n$. Since S_n is monic this implies that S_n cannot have a zero in $(-\infty,a]$. So for coherent pairs of type B all zeros of S_n are in (a,b).
However, we will see below, that for coherent pairs of type C, S_n can have a zero in $(-\infty,a)$.

(ii) The zeros of P_n are upper bounds for the zeros of S_n^{λ}. Since $S_n^0=P_n$ these upper bounds cannot be improved. Observe that for coherent pairs of type A the polynomials P_n depend on ξ and for coherent pairs of type B on M.

(iii) The zeros of P_{n-1} provide us with lower bounds for $s_2^{\lambda},\ldots,s_n^{\lambda}$ but not for s_1^{λ}.

For coherent pairs of type A Lemma 3.6 enables us to derive more information on the location of the zeros of S_n^λ.

Theorem 4.2. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A. Let $n \geqslant 2$. Then*
(i) *in the Laguerre case the zeros of S_n^λ interlace with the zeros of $L_n^{\alpha+1}$, in the Jacobi case the zeros of S_n^λ interlace with the zeros of $P_n^{\alpha,\beta+1}$; the zeros of S_n^λ are shifted to the left with respect to those of $L_n^{\alpha+1}$, respectively, $P_n^{\alpha,\beta+1}$;*
(ii) *the zeros of S_n^λ interlace with the zeros of S_n^∞; the zeros of S_n^λ are shifted to the right with respect to those of S_n^∞;*
(iii) *the $n-1$ zeros of Q_{n-1} separate the zeros of S_n^λ.*

Proof. (i) In the Laguerre case put

$$\pi(x) = \frac{x\hat{L}_n^{\alpha+1}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{L}_n^{\alpha+1}$$

in (3.9) and in the Jacobi case put

$$\pi(x) = \frac{(x+1)\hat{P}_n^{\alpha,\beta+1}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{P}_n^{\alpha,\beta+1}$$

in (3.9). Proceeding as in the proof of part (ii) of Theorem 4.1 we arrive at the desired result.

(ii) Recall from Section 2 that $S_n^\infty = \hat{L}_n^\alpha$, respectively, $\hat{P}_n^{\alpha,\beta}$.

Put

$$\pi(x) = \frac{S_n^\infty(x)}{x-\eta}, \quad \eta \text{ zero of } S_n^\infty$$

in (3.9) and apply Gauss quadrature on the zeros of S_n^∞. Then

$$S_n^\lambda(\eta)(S_n^\infty)'(\eta) < 0, \quad \eta \text{ zero of } S_n^\infty.$$

For the largest zero s_n^∞ of S_n^∞ we have $(S_n^\infty)'(s_n^\infty) > 0$, thus $S_n^\lambda(s_n^\infty) < 0$ and S_n^λ has a zero in (s_n^∞, ∞). Further we proceed as in the proof of Theorem 4.1 to obtain the desired result.

(iii) In the Laguerre case we substitute

$$\pi(x) = \frac{x\hat{L}_{n-1}^{\alpha+1}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{L}_{n-1}^{\alpha+1},$$

a polynomial of degree $n-1$ in (3.9).

In the Jacobi case we take

$$\pi(x) = \frac{(x^2-1)\hat{P}_{n-1}^{\alpha+1,\beta+1}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{P}_{n-1}^{\alpha+1,\beta+1},$$

a polynomial of degree n in (3.9). In both cases we obtain

$$\int_a^b S_n^\lambda(x)\frac{Q_{n-1}(x)}{x-\eta}w_1(x)\,\mathrm{d}x < 0, \quad \eta \text{ zero of } Q_{n-1}.$$

Then we proceed as in the proof of part (iii) of Theorem 4.1. □

4.2. Remarks on theorem 4.2

(1) Part (i) of the theorem gives upper bounds, independent of ξ, for the zeros of S_n^λ. Since for $\xi = a$ the polynomial S_n^0 equals the classical polynomials mentioned, these upper bounds cannot be improved, when we want to have bounds independent of ξ and λ.
(ii) The zeros of S_n^∞ give lower bounds for the zeros of S_n^λ. Observe that these lower bounds are independent of ξ and that they cannot be improved when we want to have bounds independent of λ. In particular part (ii) of Theorem 4.2 implies that all zeros of S_n^λ are in (a,b).
(iii) Since the bounds for the zeros of S_n^λ in (i) and (ii) are best possible (independent of ξ and λ), part (iii) of Theorem 4.2 gives no further information on the position of the zeros of S_n^λ. In fact it is a result on the location of the zeros of different classical polynomials with respect to each other.

Theorem 4.3. *Let $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ denote a coherent pair of type B. Let $n \geqslant 2$. Then*
(i) *in the Laguerre case the zeros of S_n^λ interlace with the zeros of L_n^0, in the Jacobi case the zeros of S_n^λ interlace with the zeros of $P_n^{\alpha,0}$; the zeros of S_n^λ are shifted to the left with respect to those of L_n^0, respectively, $P_n^{\alpha,0}$;*
(ii) *the $n-1$ zeros of Q_{n-1} separate the zeros of S_n^λ.*

Proof. (i) The proof is similar to the proof of part (ii) of Theorem 4.1 by using relation (3.7) instead of (3.8).

(ii) In the Laguerre case put

$$\pi(x) = \frac{\hat{L}_{n-1}^0(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{L}_{n-1}^0$$

a polynomial of degree $n-2$ in (3.7).

For the Jacobi case take

$$\pi(x) = \frac{(x-1)\hat{P}_{n-1}^{\alpha+1,0}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{P}_{n-1}^{\alpha+1,0}$$

a polynomial of degree $n-1$ in (3.7). In both cases

$$\int_a^b S_n^\lambda(x)\frac{Q_{n-1}(x)}{x-\eta}w_1(x)\,\mathrm{d}x < 0, \quad \eta \text{ zero of } Q_{n-1}.$$

Further proceed as in the proof of Theorem 4.1(iii). □

Theorem 4.3 for coherent pairs of type B correponds with Theorem 4.2(i) and (iii) for coherent pairs of type A. It is also possible to prove that for coherent pairs of type B the S_n^λ satisfy assertion (ii) of Theorem 4.2. Therefore we have to define a Gauss quadrature on the zeros of S_n^∞. It gives no special work to define it for arbitrary coherent pairs.

Lemma 4.1. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A, B or C. Let $n \geqslant 2$. Let $s_1^\infty < s_2^\infty < \cdots < s_n^\infty$ denote the zeros of S_n^∞.*
If f is a polynomial of degree $\leqslant 2n-2$ then

$$\int_{-\infty}^{\infty} f(x)\,d\psi_0(x) = \sum_{k=1}^{n} f(s_k^\infty)\lambda_k, \tag{4.1}$$

where the λ_k are positive and independent of f.

Proof. Let $l_1(x), \ldots, l_n(x)$ denote the Lagrange interpolation polynomials of degree $n-1$ on the zeros of S_n^∞, i.e.

$$l_k(s_j^\infty) = \begin{cases} 0 & j \neq k, \\ 1 & j = k. \end{cases}$$

Write

$$f(x) = \sum_{k=1}^{n} f(s_k^\infty) l_k(x) + S_n^\infty(x) r(x),$$

where r is a polynomial of degree $\leqslant n-2$.
Then

$$\int_{-\infty}^{\infty} f(x)\,d\psi_0(x) = \sum_{k=1}^{n} f(s_k^\infty) \int_{-\infty}^{\infty} l_k(x)\,d\psi_0(x) + \int_{-\infty}^{\infty} S_n^\infty(x) r(x)\,d\psi_0(x).$$

We use (2.8) to obtain that the last integral equals zero.
Hence we have proved (4.1) with

$$\lambda_k = \int_{-\infty}^{\infty} l_k(x)\,d\psi_0(x),$$

independent of f. To obtain that λ_k is positive, substitute in (4.1)

$$f(x) = \left(\frac{S_n^\infty(x)}{x - s_k^\infty} \right)^2. \qquad \square$$

Theorem 4.4. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type B. Let $n \geqslant 2$. Then the zeros of S_n^λ interlace with the zeros of S_n^∞; the zeros of S_n^λ are shifted to the right with respect to the zeros of S_n^∞.*

Proof. Recall from Section 2,

$$S_n^\infty(x) = x\hat{L}_{n-1}^1(x) \quad \text{in the Laguerre case,}$$

$$S_n^\infty(x) = (x+1)\hat{P}_{n-1}^{\alpha,1}(x) \quad \text{in the Jacobi case.}$$

Hence the smallest zero s_1^∞ of S_n^∞ is $x = a$.

Put in (4.1)

$$f(x) = S_n^\lambda(x)\frac{S_n^\infty(x)}{(x-a)(x-\eta)},$$

where $\eta = s_k^\infty$ denotes a zero of S_n^∞ with $\eta \neq a$. Then

$$\int_{-\infty}^{\infty} S_n^\lambda(x)\frac{S_n^\infty(x)}{(x-a)(x-\eta)}\,\mathrm{d}\psi_0(x) = \lambda_1\frac{S_n^\lambda(a)(S_n^\infty)'(a)}{a-\eta} + \lambda_k\frac{S_n^\lambda(\eta)(S_n^\infty)'(\eta)}{\eta-a}. \tag{4.2}$$

By Lemma 3.5 the integral in (4.2) is negative. Consider the first term on the right-hand side of (4.2). By Lemma 3.3 we have $\operatorname{sgn} S_n^\lambda(a) = (-1)^n$. Moreover $\operatorname{sgn}(S_n^\infty)'(a) = (-1)^{n-1}$ and $a - \eta < 0$. Then the first term in the right-hand side of (4.2) is positive. Since the left-hand side of (4.2) is negative, the second term in the right-hand side has to be negative. Then

$$S_n^\lambda(s_k^\infty)(S_n^\infty)'(s_k^\infty) < 0$$

for all zeros s_k^∞ of S_n^∞, including $s_1^\infty = a$. This implies that each of the n intervals $(s_1^\infty, s_2^\infty), \ldots, (s_{n-1}^\infty, s_n^\infty), (s_n^\infty, \infty)$ contains a zero of S_n^λ. □

Theorem 4.5. *Let $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ denote a coherent pair of type A or B. Let $n \geqslant 2$. Then the $n-1$ zeros of S_{n-1}^λ separate the zeros of S_n^λ.*

Proof. By Theorem 4.2(ii) for type A and Theorem 4.4 for type B, the zeros of S_n^λ interlace with those of S_n^∞. Then in two consecutive zeros of S_n^λ the polynomial S_n^∞ has opposite sign. Relation (2.9)

$$\frac{S_n^\infty}{n} = \frac{S_n^\lambda}{n} + \mathrm{d}_{n-1}\frac{S_{n-1}^\lambda}{n-1}, \quad n \geqslant 2,$$

implies that in a zero of S_n^λ the polynomials S_n^∞ and S_{n-1}^λ have the same sign. Then in two consecutive zeros of S_n^λ the polynomial S_{n-1}^λ has opposite sign. This implies the assertion of the theorem. □

5. Coherent pairs of type C

In Theorem 4.1 we have mentioned all the location properties for which we can prove that they are satisfied by S_n^λ for an arbitrary coherent pair. In Theorems 4.2, 4.3 and 4.5 we have shown that for coherent pairs of type A and B the polynomials $S_n^\lambda (n \geqslant 2)$ moreover satisfy the following assertions:

1. the zeros of S_n^λ interlace with the zeros of S_n^∞; the zeros of S_n^λ are shifted to the right with respect to those of S_n^∞;
2. the $n-1$ zeros of S_{n-1}^λ separate the zeros of S_n^λ;
3. the $n-1$ zeros of Q_{n-1} separate the zeros of S_n^λ.

In the present section we will show that these assertions, in general, are not satisfied by coherent pairs of type C. By lack of the first assertion we are missing a lower bound for the smallest zero of $S_n^\lambda (n \geqslant 2)$. Therefore such a lower bound is derived in Theorem 5.2.

We start with a special situation, the Laguerre case with $\xi = 0$, $M = 0$, also studied in [2].

Lemma 5.1. *Let* $\mathrm{d}\psi_0 = \mathrm{d}\psi_1 = x^\alpha e^{-x}\,\mathrm{d}x$ *where* $\alpha > -1$. *Then*

$$S_n^\infty(x) = x\hat{L}_{n-1}^\alpha(x) - (n+\alpha-1)S_{n-1}^\infty, \quad n \geqslant 3, \tag{5.1}$$

$$S_2^\infty(x) = x^2 - 2(\alpha+1)x + \alpha(\alpha+1). \tag{5.2}$$

Proof. From the well-known relation for monic Laguerre polynomials, compare e.g. Szegő [6], p. 102,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\hat{L}_n^\alpha(x) + n\hat{L}_{n-1}^\alpha(x)\} = n\hat{L}_{n-1}^\alpha(x)$$

together with (2.10) and (2.11) we obtain

$$S_n^\infty = \hat{L}_n^\alpha + n\hat{L}_{n-1}^\alpha, \quad n \geqslant 2. \tag{5.3}$$

The recurrence relation for monic Laguerre polynomials reads

$$\hat{L}_n^\alpha(x) = (x - 2n - \alpha + 1)\hat{L}_{n-1}^\alpha(x) - (n-1)(n+\alpha-1)\hat{L}_{n-2}^\alpha(x).$$

Relation (5.1) is a direct consequence of the recurrence relation and (5.3); (5.2) follows from a simple calculation using (5.3). □

Theorem 5.1. *Let* $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ *denote the coherent pair of type C with* $\mathrm{d}\psi_0 = \mathrm{d}\psi_1 = x^\alpha e^{-x}\,\mathrm{d}x$, *where* $-1 < \alpha < 0$. *Let* s_{1n}^∞ *denote the smallest zero of* S_n^∞ *and* s_{1n}^λ *the smallest zero of* S_n^λ. *Then*

(i) $s_{1n}^\infty < 0$ *if* $n \geqslant 2$;
(ii) s_{12}^∞ *is a lower bound for the zeros of* S_n^λ *for all* $n \geqslant 1$ *and all* $\lambda > 0$;
(iii) *if* $n \geqslant 3$, *then there exist* (*large*) λ *such that* $s_{1,n-1}^\lambda < s_{1n}^\lambda < s_{1n}^\infty$ *and there exist* (*small*) λ *such that* $s_{1n}^\infty < s_{1n}^\lambda < s_{1,n-1}^\lambda$.

Proof. (i) Substitute $x = 0$ in (5.1) and (5.2) then

$$S_2^\infty(0) = \alpha(\alpha+1) < 0,$$

$$sgn\, S_n^\infty(0) = -\, sgn\, S_{n-1}^\infty(0), \quad n \geqslant 3.$$

Hence $sgn\, S_n^{\infty}(0) = (-1)^{n-1}$, $n \geqslant 2$. Since S_n^{∞} is monic this implies that S_n^{∞} has at least one zero in $(-\infty, 0)$. Observe that Theorem 4.1(i) implies that S_n^{∞} $(n \geqslant 2)$ has exactly one negative zero.

(ii) Put $x_0 = s_{12}^{\infty} < 0$. Then $sgn\,\{x_0 \hat{L}_{n-1}^{\alpha}(x_0)\} = (-1)^n$. Using (5.1) we obtain

$$sgn\, S_3^{\infty}(x_0) = -1.$$

Suppose $sgn\, S_{n-1}^{\infty}(x_0) = (-1)^{n-1}$ for some $n \geqslant 4$, then (5.1) implies $sgn\, S_n^{\infty}(x_0) = (-1)^n$. Hence

$$sgn\, S_n^{\infty}(x_0) = (-1)^n \quad \text{for all } n \geqslant 3. \tag{5.4}$$

This implies that all zeros of S_n^{∞} $(n \geqslant 3)$ are to the right of x_0. To obtain a similar result for the zeros of S_n^{λ} we use (2.9)

$$\frac{S_n^{\infty}}{n} = \frac{S_n^{\lambda}}{n} + d_{n-1}\frac{S_{n-1}^{\lambda}}{n-1}, \quad \text{where } d_{n-1} > 0,\ n \geqslant 2. \tag{5.5}$$

Recall $S_1^{\lambda} = P_1$, so $S_1^{\lambda}(x_0) < 0$.

Then (5.5) with $n = 2$ gives $S_2^{\lambda}(x_0) > 0$. As before we proceed by induction. Suppose that for some $n \geqslant 3$ we have $sgn\, S_{n-1}^{\lambda}(x_0) = (-1)^{n-1}$. Then (5.4) and (5.5) give $sgn\, S_n^{\lambda}(x_0) = (-1)^n$.

Hence

$$sgn\, S_n^{\lambda}(x_0) = (-1)^n \quad \text{for all } n \geqslant 1 \text{ and all } \lambda > 0.$$

Since S_n^{λ} has at most one negative zero this implies that all zeros of S_n^{λ} are to the right of x_0.

(iii) Let $n \geqslant 3$. Substitute the negative zero $s_{1,n-1}^{\infty}$ of S_{n-1}^{∞} in (5.1). We obtain

$$sgn\, S_n^{\infty}(s_{1,n-1}^{\infty}) = (-1)^n.$$

This implies that the negative zero s_{1n}^{∞} of S_n^{∞} is to the right of $s_{1,n-1}^{\infty}$:

$$s_{1,n-1}^{\infty} < s_{1,n}^{\infty} < 0.$$

If λ is sufficiently large S_{n-1}^{λ} and S_n^{λ} have a negative zero $s_{1,n-1}^{\lambda}$, respectively $s_{1,n}^{\lambda}$ such that

$$s_{1,n-1}^{\lambda} < s_{1n}^{\lambda} < 0.$$

This implies $sgn\, S_{n-1}^{\lambda}(s_{1n}^{\lambda}) = (-1)^n$. Again we use (5.5) and obtain

$$sgn\, S_n^{\infty}(s_{1n}^{\lambda}) = (-1)^n,$$

i.e. the negative zero of S_n^{∞} is to the right of s_{1n}^{λ}.

On the other hand $S_n^0 = P_n$, $S_{n-1}^0 = P_{n-1}$, so if λ is sufficiently small the smallest zeros of S_n^{λ} and S_{n-1}^{λ} are in $(0, \infty)$, i.e. to the right of the negative zero of S_n^{∞} and $s_{1n}^{\lambda} < s_{1,n-1}^{\lambda}$. $\square$

5.1. Remark on Theorem 5.1

Part (iii) of the theorem implies that we cannot make a general assertion on the position of the zeros of S_n^{λ} with respect to those of S_n^{∞}. Moreover if $s_{1,n-1}^{\lambda} < s_{1n}^{\lambda}$ the $n-1$ zeros of S_{n-1}^{λ} cannot separate the n zeros of S_n^{λ}. In particular, since $s_{1,n-1}^{\lambda}$ is a continuous function of λ, there has to exist (at least) one λ such that the negative zeros of S_n^{∞}, S_n^{λ} and S_{n-1}^{λ} coincide.

It is our intention to derive a lower bound for the zeros of S_n^{λ} for all $n \geqslant 1$ and all $\lambda > 0$ for an arbitrary coherent pair of type C. We will do it by generalization of the method of Theorem 5.1(ii).

Lemma 5.2. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type A, B or C. Let $\{P_n\}$ denote the MOPS with respect to $d\psi_0$. Then $\{S_n^{\infty}\}$ satisfies a recurrence relation of the form*

$$S_n^{\infty}(x) = (x - \xi + \eta_n)P_{n-1}(x) - h_n S_{n-1}^{\infty}(x), \quad n \geqslant 3, \tag{5.6}$$

$$S_2^{\infty}(x) = (x - \xi + \eta_2)P_1(x) - h_2, \tag{5.7}$$

where $h_n > 0$, $n \geqslant 2$; in particular $h_2 = \frac{\|P_1\|^2}{\|P_0\|^2}$.

Proof. It is well-known that $\{P_n\}$ satisfies a three term recurrence relation

$$P_n(x) = (x + B_n)P_{n-1}(x) - C_n P_{n-2}(x), \quad n \geqslant 2$$

with

$$C_n = \frac{\|P_{n-1}\|^2}{\|P_{n-2}\|^2} > 0.$$

On the other hand by (2.8)

$$S_n^{\infty} = P_n + D_{n-1}\frac{n}{n-1}P_{n-1}, \quad n \geqslant 2.$$

Then, for $n \geqslant 3$

$$S_n^{\infty} = \left(x + B_n + D_{n-1}\frac{n}{n-1} + \frac{n-2}{n-1}\frac{C_n}{D_{n-2}}\right)P_{n-1} - \frac{n-2}{n-1}\frac{C_n}{D_{n-2}}\left(P_{n-1} + D_{n-2}\frac{n-1}{n-2}P_{n-2}\right)$$

and (5.6) follows.

For $n = 2$ we have

$$S_2^{\infty} = (x + B_2 + 2D_1)P_1 - C_2 P_0$$

and (5.7) follows. $\square$

Lemma 5.3. *Let $\{d\psi_0, d\psi_1\}$ denote a coherent pair of type C. Let $\{P_n\}$ be the MOPS with respect to $d\psi_0$. Then the η_n in (5.6) and (5.7) satisfy*

$$\eta_n < -P_1(\xi), \quad n \geqslant 2.$$

Proof. Differentiate (5.7), apply (2.11) and write $P_1(x)=P_1(\xi)+(x-\xi)$, then

$$2Q_1(x)=P_1(\xi)+\eta_2+2(x-\xi).$$

From $\int Q_1(x)\,\mathrm{d}\psi_1(x)=0$ we obtain

$$(P_1(\xi)+\eta_2)\int \mathrm{d}\psi_1(x)+2\int (x-\xi)\,\mathrm{d}\psi_1(x)=0. \tag{5.8}$$

Since the integrals are positive, this implies

$$P_1(\xi)+\eta_2<0,$$

which proves the lemma for $n=2$. Observe that the first term in (5.8) depends on M and the second one is independent of M. So if $M\to\infty$, then $\eta_2\to -P_1(\xi)$, and the upper bound for η_2 cannot be improved.

Let $n\geqslant 3$. Differentiation of (5.6) gives

$$nQ_{n-1}(x)=P_{n-1}(x)+(x-\xi)P'_{n-1}(x)+\eta_n P'_{n-1}(x)-h_n(n-1)Q_{n-2}(x).$$

Then

$$\int P_{n-1}\,\mathrm{d}\psi_1+\int (x-\xi)P'_{n-1}\,\mathrm{d}\psi_1+\eta_n\int P'_{n-1}\,\mathrm{d}\psi_1=0.$$

In the Laguerre case $P'_{n-1}=(n-1)\hat{L}_{n-2}^{\alpha+1}$ and in the Jacobi case $P'_{n-1}=(n-1)\hat{P}_{n-2}^{\alpha+1,\beta+1}$. In both cases the second integral becomes zero.

Hence

$$\eta_n(-1)^n\int P'_{n-1}\,\mathrm{d}\psi_1=(-1)^{n-1}\int P_{n-1}\,\mathrm{d}\psi_1,\quad n\geqslant 3. \tag{5.9}$$

The left-hand side of (5.9) is evaluated by

$$(-1)^n\int P'_{n-1}\,\mathrm{d}\psi_1=(-1)^n\int P'_{n-1}w_1\,\mathrm{d}x+(-1)^nMP'_{n-1}(\xi). \tag{5.10}$$

Observe that the last term is $\geqslant 0$. The preceding term is positive, which can be seen by applying the Rodrigues formula and integration by parts. In the Laguerre case the calculation is:

$$\begin{aligned}(-1)^n\int P'_{n-1}w_1\,\mathrm{d}x&=(-1)^n(n-1)\int_0^\infty \hat{L}_{n-2}^{\alpha+1}(x)\frac{x^{\alpha+1}e^{-x}}{x-\xi}\,\mathrm{d}x\\ &=(n-1)\int_0^\infty \frac{1}{x-\xi}D^{n-2}(e^{-x}x^{n+\alpha-1})\,\mathrm{d}x=(n-1)!\int_0^\infty \frac{e^{-x}x^{n+\alpha-1}}{(x-\xi)^{n-1}}\,\mathrm{d}x>0.\end{aligned}$$

A similar calculation can be made in the Jacobi case.

The right-hand side of (5.9) is evaluated by

$$(-1)^{n-1}\int P_{n-1}\,\mathrm{d}\psi_1=(-1)^{n-1}\int P_{n-1}w_1\,\mathrm{d}x+(-1)^{n-1}MP_{n-1}(\xi). \tag{5.11}$$

Again the last term is $\geqslant 0$. But now the first term in the right-hand side is $\leqslant 0$. In the Laguerre case the calculation reads as follows, where we use $x=(x-\xi)+\xi$:

$$(-1)^{n-1}\int P_{n-1}w_1\,\mathrm{d}x=(-1)^{n-1}\int_0^\infty \hat{L}^\alpha_{n-1}(x)\frac{x^{\alpha+1}e^{-x}}{x-\xi}\,\mathrm{d}x$$
$$=\xi(-1)^{n-1}\int_0^\infty \hat{L}^\alpha_{n-1}(x)\frac{x^{\alpha}e^{-x}}{x-\xi}\,\mathrm{d}x.$$

As above the sign of the last integral is $(-1)^{n-1}$. Since $\xi\leqslant 0$, the last term is $\leqslant 0$. In the Jacobi case we can make a similar calculation with the substitution $1-x^2=-(x-\xi)^2-2\xi(x-\xi)+(1-\xi^2)$.

We obtain from (5.9), (5.10), (5.11) and the observations on the signs of the terms:
if $M=0$, then $\eta_n\leqslant 0$,
if $M>0$, then

$$\eta_n=\frac{(-1)^{n-1}\int P_{n-1}\,\mathrm{d}\psi_1}{(-1)^n\int P'_{n-1}\,\mathrm{d}\psi_1}\leqslant-\frac{P_{n-1}(\xi)}{P'_{n-1}(\xi)}. \tag{5.12}$$

Since the right-hand side of (5.12) is positive the estimation (5.12) holds for all coherent pairs of type C.

By the Christoffel–Darboux relation

$$P'_{n-1}(\xi)P_{n-2}(\xi)-P'_{n-2}(\xi)P_{n-1}(\xi)>0,\quad n\geqslant 2.$$

Since $P'_{n-1}(\xi)P'_{n-2}(\xi)<0$ this implies

$$-\frac{P_{n-1}(\xi)}{P'_{n-1}(\xi)}<-\frac{P_{n-2}(\xi)}{P'_{n-2}(\xi)}.$$

By repeated application (5.12) implies

$$\eta_n<-P_1(\xi),\quad n\geqslant 3.$$

This proves the lemma. □

Theorem 5.2. *Let* $\{\mathrm{d}\psi_0,\mathrm{d}\psi_1\}$ *denote a coherent pair of type C. Let* $\{P_n\}$ *denote the MOPS with respect to* $\mathrm{d}\psi_0$. *Put* $x_0=\xi+P_1(\xi)-\frac{1}{2}$. *Then* x_0 *is a lower bound for the zeros of* S_n^λ *for all* $n\geqslant 1$ *and all* $\lambda>0$.

Proof. First we prove $S_2^\infty(x_0)>0$. With Lemma 5.3 we have $x_0-\xi+\eta_2<-\frac{1}{2}$. Since $x_0<\xi\leqslant a$ we also have $P_1(x_0)<0$.

Hence, with (5.7),

$$S_2^\infty(x_0)>-\frac{1}{2}P_1(x_0)-h_2.$$

Writing $P_1(x)=(x-\xi)+P_1(\xi)$ we obtain $P_1(x_0)=2P_1(\xi)-\frac{1}{2}$, thus

$$S_2^{\infty}(x_0) > -P_1(\xi)+\frac{1}{4}-h_2.$$

In the Laguerre case we have (see Szegő [6]) $P_1(\xi)=\xi-\alpha-1$, $h_2=\alpha+1$; so $S_2^{\infty}(x_0)>0$.

In the Jacobi case

$$P_1(\xi)=\xi+\frac{\alpha-\beta}{\alpha+\beta+2}, \quad h_2=\frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}.$$

A tedious calculation shows that also in this case $S_2^{\infty}(x_0)>0$. (The calculation can be simplified by substituting $\alpha+1=\alpha_1>0$, $\beta+1=\beta_1>0$.)

Further we proceed as in the proof of Theorem 5.1(ii). First we prove by induction, starting from $S_2^{\infty}(x_0)>0$ and using (5.6),

$$sgn\, S_n^{\infty}(x_0)=(-1)^n, \quad n\geqslant 2.$$

Then we apply (5.5). The starting value $sgn\, S_1^{\lambda}(x_0)=-1$ and induction gives

$$sgn\, S_n^{\lambda}(x_0)=(-1)^n \quad \text{for all } \lambda>0.$$

Since by Theorem 4.1(i) the monic polynomial S_n^{λ} can have at most one zero in $(-\infty,a)$ and $x_0<a$, this implies that all zeros of S_n^{λ} are to the right of x_0. □

5.2. *Zeros of Q_{n-1}*

By (2.11) we have for $n\geqslant 2$ that $(S_n^{\infty})'=nQ_{n-1}$, thus the zeros of Q_{n-1} are just the extremata of S_n^{∞}. Then the zeros of Q_{n-1} separate the zeros of S_n^{∞}, i.e. the zeros of Q_{n-1} separate the zeros of S_n^{λ} if λ is *sufficiently large*. In Theorems 4.2(iii) and 4.3(ii) we have seen that for coherent pairs of type A and B this property holds for all λ. This however is not the case for all coherent pairs of type C.

As an example we take the Laguerre case with $n=2$. Then $S_2^0=\hat{L}_2^{\alpha}$, i.e. if λ is small, then the zeros of S_2^{λ} are close to the zeros of $\hat{L}_2^{\alpha}$.

Put $Q_1(x)=x-q$, where q depends on M and ξ. From $\int Q_1\,\mathrm{d}\psi_1=0$ it follows

$$q\left\{M+\int_0^{\infty}\frac{x^{\alpha+1}e^{-x}}{x-\xi}\,\mathrm{d}x\right\}=\int_0^{\infty}\frac{x^{\alpha+2}e^{-x}}{x-\xi}\,\mathrm{d}x+M\xi.$$

Keep ξ fixed and let $M\to\infty$, then $q\to\xi\leqslant 0$, i.e. the δ-function in ξ attracts the zero q of Q_1. Thus if M is sufficiently large and λ is sufficiently small the zero q of Q_1 cannot separate the two zeros of S_2^{λ}.

We will show however that if $M=0$ then the zeros of Q_{n-1} still separate the zeros of S_n^{λ}.

Lemma 5.4. *Let $\{\mathrm{d}\psi_0,\mathrm{d}\psi_1\}$ denote a coherent pair of type A, B or C. Let $n\geqslant 2$. Then the $n-1$ extremata of S_n^{λ} interlace with the $n-1$ zeros of Q_{n-1}; the extremata of S_n^{λ} are shifted to the right with respect to the zeros of Q_{n-1}.*

Proof. Put

$$\pi(x) = \frac{Q_{n-1}(x)}{x-\eta},$$

where η denotes a zero of Q_{n-1}. Then $\pi(x)$ is a polynomial of degree $n-2$ with $n-2$ real zeros $> \xi$. Similar to the proof of Lemma 3.5 we denote the zeros by $\xi + t_1, \dots, \xi + t_{n-2}$ where $t_1, \dots, t_{n-2}$ are positive. Then

$$\pi(x) = (x-\xi-t_1)\cdots(x-\xi-t_{n-2}) = \sum_{i=0}^{n-2} c_i(x-\xi)^i,$$

where $c_{n-2} = 1$ and $sgn\, c_i = (-1)^{n-i}$.

Using the notation defined in (3.1) we obtain

$$I = \int_{-\infty}^{\infty} S_n'(x) \frac{Q_{n-1}(x)}{x-\eta}\, \mathrm{d}\psi_1(x) = \sum_{i=0}^{n-2} c_i \mu_i^n.$$

For $0 \leqslant i \leqslant n-2$ we have by (3.2) and Lemma 3.4

$$sgn\, \mu_i^n = -\, sgn\, m_{i+1}^n = (-1)^{n+i-1}.$$

Hence

$$I = \int_{-\infty}^{\infty} S_n'(x) \frac{Q_{n-1}(x)}{x-\eta}\, \mathrm{d}\psi_1(x) < 0.$$

With Gauss quadrature on the zeros of Q_{n-1} the result follows. □

Theorem 5.3. *Let $\{\mathrm{d}\psi_0, \mathrm{d}\psi_1\}$ denote a coherent pair of type C with $M = 0$, let $n \geqslant 2$. Then the $n-1$ zeros of Q_{n-1} separate the zeros of S_n^λ.*

Proof. We use an auxiliary polynomial H_{n-1} of degree $n-1$. In the Laguerre case $H_{n-1} = L_{n-1}^\alpha$ and in the Jacobi case $H_{n-1} = P_{n-1}^{\alpha+1,\beta}$.

The $n-1$ zeros of H_{n-1} separate the n zeros of S_n^λ. In the Laguerre case this follows from Theorem 4.1(iii) since $H_{n-1} = L_{n-1}^\alpha = P_{n-1}$. In the Jacobi case it can be proved by substituting

$$\pi(x) = \frac{(x-1)\hat{P}_{n-1}^{\alpha+1,\beta}(x)}{x-\eta}, \quad \eta \text{ zero of } \hat{P}_{n-1}^{\alpha+1,\beta}$$

in (3.7) and applying Gauss quadrature with remainder term on the zeros of $P_{n-1}^{\alpha+1,\beta}$ (compare the proof of Theorem 4.1(iii)).

Thus if $s_1 < s_2 < \cdots < s_n$ denote the zeros of S_n^λ and $h_1 < h_2 < \cdots < h_{n-1}$ the zeros of H_{n-1}, then

$$s_i < h_i < s_{i+1} \quad i = 1, \dots, n-1. \tag{5.13}$$

Let $w_H(x)$ denote the weight-function corresponding to H_{n-1}, i.e. $w_H(x) = x^\alpha e^{-x}$ in the Laguerre case and $w_H(x) = (1-x)^{\alpha+1}(1-x)^\beta$ in the Jacobi case. Then

$$\frac{w_1(x)}{w_H(x)} = \begin{cases} \dfrac{x}{x-\xi} & \text{Laguerre case,} \\ \dfrac{1+x}{x-\xi} & \text{Jacobi case.} \end{cases}$$

Thus $w_1(x)/w_{H(x)}$ is an increasing function in x on (a,b) if $\xi \neq a$. This implies that if $q_1 < q_2 < \cdots < q_{n-1}$ denote the zeros of Q_{n-1}, then

$$h_i \leqslant q_i, \quad i = 1,\dots,n-1 \tag{5.14}$$

(see Szegő [6], p. 116).

Finally let $\sigma_1 < \sigma_2 < \cdots < \sigma_{n-1}$ denote the extremata of S_n^λ. By Lemma 5.4

$$q_i < \sigma_i, \quad i = 1,\dots,n-1. \tag{5.15}$$

Obviously

$$s_i < \sigma_i < s_{i+1}, \quad i = 1,\dots,n-1. \tag{5.16}$$

Then from (5.13), (5.14), (5.15) and (5.16) we obtain

$$s_i < h_i \leqslant q_i < \sigma_i < s_{i+1}, \quad i = 1,\dots,n-1.$$

This proves the theorem. □

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